That Weird Math Course

O. Guy Morley

February 1, 2019

If there is a choice between a test and no test, are you going to choose a test? Don't you have bad memories of tests? I had more than plenty. So, when I had to choose a section of a required math course in college, I automatically chose the one with no test. Prof. Matt Fovia was well known for that. Yay! That was a relief. I also knew that he gave a lot of homework and that course was not easy at all. But those seemed a small price for avoiding tests.

I was a sophomore, majoring in Computer Science. And, Discrete Mathematics was a required course. Actually, it was called Discrete Structures of Computer Science at my college. Any way, to most of us, the course seemed extremely abstract and dry. We didn't know why it was required for the first place. On the first day, Prof. Fovia told us about the course. Disc Math is a set of components to model the real world ... mathematically and thus computationally. Since we, computer science students, were learning to develop computer systems and programs that deal with the real world, it was supposed to be absolutely essential. He also said that it is pointless if he couldn't convince us early in the semester. That was his main concern. I was curious how he could ever do that.

Soon after the course introduction, Prof. Fovia gave us the very first in-class exercise. We were told to work in pairs of students, assigned by him. The task was to come up with a drawing of a (hypothetical) North Pole scenario satisfying the conditions shown below. Note that texts in gray background (throughout this essay) are directly from Prof. Fovia's materials.

Exercise 1: North Pole

1. Reindeer are not Santa Claus.

2. Reindeer must carry someone/something.
3. Santa Claus must be carried by reindeer.
4. A reindeer exists.

So, my partner, Eric, and I worked on the task. We decided to take turn to complete the drawing. I drew a reindeer in the middle. Eric drew seven more reindeer. I drew a sleigh drawn by the eight reindeer. Eric drew Santa on the sleigh. Then, we checked the Conditions one by one. I pointed out that we didn't need eight reindeer. But Eric said that that does not violate the Conditions either. So, we were settled. Finally, Eric added the scenery. That too seemed all right with the Conditions. The final drawing looked like this.

Then, All the pairs of students shared their drawings. Most of them were rather similar to ours. Most pairs had eight reindeer but some groups had different numbers. We all agreed that the number of reindeer did not matter.

However, when we saw one pair's drawing, we were stunned. In their drawing, a reindeer was carrying another reindeer. According to the pair, the Conditions do not exclude the possibility that there be no Santa. Since a reindeer must carry "something," they chose another reindeer for that purpose. To this, Eric commented that Condition 3 seemed to assume that there is Santa. The class were equally divided. Then, Prof. Fovia intervened. He said that the English language may not be sufficiently specific about the situation. He also said that we would learn a more precise way of representing conditions. He was pleased that the class took the exercise seriously and examined various possibilities.

After this, Prof. Fovia asked us the following question. Would the Conditions be satisfied by a drawing of a reindeer carrying *itself*? Again, the class had an active discussion without any conclusion. I personally thought that it was possible too. The Conditions seemed to have a lot of "loopholes."

At the end of the class, Prof. Fovia explained that this exercise was to realize the connection between conditions and drawings. The connection can also be understood as the one between conditions (or specification) and "scenarios" (constrained by the specification). He said that this kind of connection is the essence underlying modeling the real world. According to him, if we learn how to do this precisely and concisely, we can gain a better understanding of computational modeling.

The first class meeting was all right for me. I don't particularly like math. But the discussion was more like solving a puzzle. I didn't have to juggle symbols, as in many other math courses. Then, Prof. Fovia encouraged us to be creative. He also said that we should not be stuck to his expectations either. The most surprising thing he said was that his undergraduate major was "jewelry making." It didn't make sense to me at the time. But he said we might get it by the end of the course.

The class meetings were pretty much discussion of in-class exercises done in pairs. Prof. Fovia created different pairs every time. This was for us to interact with all the other students in class. I also heard that he was using a computer program to create those pairs. And for each class meeting, there was homework. The homework was either some extension of discussions or preparation for the next class. These were tedious. Some seemed pointless and some seemed mind-boggling. Any way, it was not at all like any other math courses.

ππππππππππππππππππππππππππππππππππππππ

Prof. Fovia wanted us to be really familiar with the connection between conditions and scenarios. So, he introduced many more exercises like the **North Pole** exercise. Here is another one.

Exercise 2: Objects in a Room

Consider the following conditions about objects in a room:

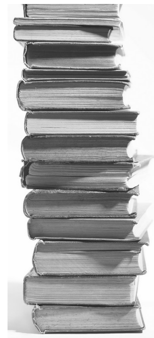
1. An object must have another object on top of it.
2. An object cannot be on top of itself.
3. If an object *X* is on top of another object *Y*, *Y* cannot be on top of *X*.

Then, consider the following five cases and respond to the associated questions.

Case A: The room does not have a ceiling. There is an object (call it Object #1) on the floor. There is another object (call it Object #2) on top of Object #1. There is yet another object (call it Object #3) on top of Object #2. More objects are stacked in this manner *without limit*.

Question: Examine whether this scenario is consistent with the three conditions. In other words, is any of the three conditions violated?

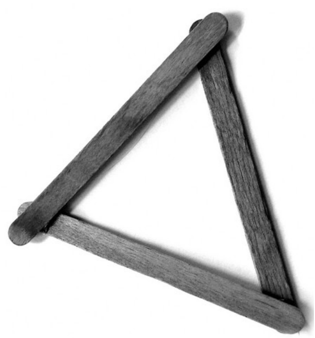
My response was that this scenario is consistent with the three conditions. There is no violation.



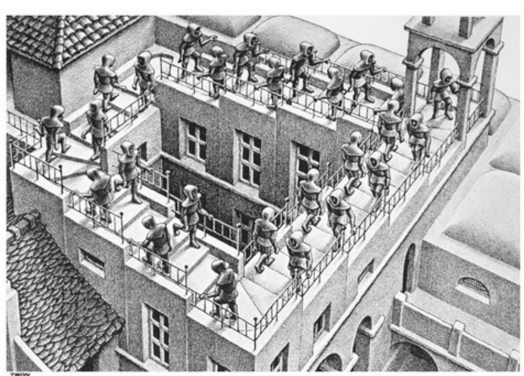
Case B: There are three objects.

Question: Would it be possible to satisfy all of the three conditions? Explain.

It wasn't obvious at first. But recalling Prof. Fovia's suggestion to be creative, I explored carefully. Well, I finally came up with an idea like the following image.



When an object is on top of another object, it doesn't really need to be on top completely. Well, there were some students who didn't come up with such an idea. But many did. In fact, that was what Prof. Fovia had in mind as well. He showed us a picture called *Ascending and Descending* by M.C. Escher, included below.



Of course, it is not realistic. But Prof. Fovia said that the connection between conditions and scenarios can open up to a lot of interesting, often eye-opening, possibilities.

Case C: There are two objects.

Question: Would it be possible to satisfy all of the three conditions? Explain.

This seemed impossible, even if the objects are *flexible* sticks. Suppose that we create a mutually stacking situation like earlier, with only two flexible sticks. But this would violate Condition 3.

Case D: There is one object.

Question: Would it be possible to satisfy all of the three conditions? Explain.

This is impossible. The object cannot have another (distinct) object on top of it, violating Condition 1.

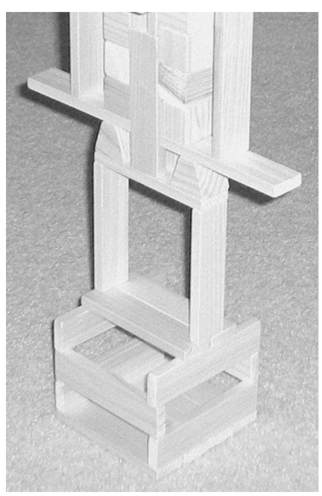
Case E: There are no objects.

Question: Would it be possible to satisfy all of the three conditions? Explain.

I wasn't sure about this question. When we reviewed this question later, many other classmates pointed out that it is possible. Just like the **North Pole** exercise, the non-existence of objects does not seem to violate any of the three conditions. So, maybe all right.

Case F/Question: Are there any other possibilities? Explain.

I thought that there could be multiple objects on top of a single object. Then, a structure like a tree could still satisfy the Conditions. Again during the review session, my classmates presented a lot of weird scenarios.



I realized that it is actually pretty hard to precisely specify a certain specific scenario with conditions. In most cases, there will be many, many unintended scenarios. Although I had no idea until then, Prof. Fovia told us that this kind of conditions are used by mathematicians to define "numbers." According to him, even with mathematically-precise conditions, there will be often infinitely-many unintended scenarios. That's kind of surprising. At the end, Prof. Fovia congratulated us for touching upon the foundation of mathematics. So, we were indeed working on math problems.

ππππππππππππππππππππππππππππππππππππππ

The previous exercise was rather abstract; Prof. Fovia admitted it. So, he gave us another one, which sounded more "realistic."

Exercise 3: Professionals

Let us consider a real-world (?) situation involving professionals. The conditions are as follows:

1. Everyone is mad.
2. There is at least one doctor.
3. There are at least two lawyers.
4. Doctors are not lawyers.
5. Lawyers sue everyone.
6. Doctors sue back if they are sued.
7. There is an individual who does not sue.

In this exercise, consider the following components:

- List of involved people (give arbitrary names to distinguish them)
- List of doctor(s), lawyer(s), and mad person/people
- Information about who sues whom

Find a scenario that is consistent with all the conditions and involves the *smallest* number of individuals. From now on, unless otherwise stated, we will focus on the smallest scenario.

I started from the minimal requirement: one doctor, Dawn (Condition 2), and two lawyers, Larry and Lauren (Condition 3). I tried to see if these three guys can satisfy the Conditions. First, they are all mad (Condition 1). Dawn is not a lawyer (Condition 4). Larry sues Dawn and Lauren, and maybe himself; Lauren sues Dawn and Larry, and again, maybe herself (Condition 5). Dawn sues Larry and Lauren (Condition 6). At this point, everyone sues someone and thus Condition 7 is not satisfied. Any additional doctor or lawyer would sue someone. So, that wouldn't help. One possibility would be to add a non-professional, say, Nora. She is still mad (Condition 1). She gets sued by Larry and Lauren (Condition 5). But since she is not a doctor, she does not need to sue anyone. This must be the smallest scenario that satisfies the Conditions.

- Doctor: Dawn
- Lawyer: Larry, Lauren
- Mad people: Dawn, Larry, Lauren, Nora
- Dawn sues Larry and Lauren.
- Larry sues Lauren, Dawn, Nora, and himself.
- Lauren sues Larry, Dawn, Nora, and herself.
- Nora doesn't sue anyone.

I was getting the idea. Of course, this exercise involves an absurd set of conditions to specify an absurd scenario. I thought that there be a way to specify the same scenario in a different way. For example, adding a condition, "people who are neither a doctor or a lawyer doesn't sue anyone" does not affect the way scenarios are satisfied. Such a condition will facilitate the existence of non-professionals more easily.

During Module A of the course (about a quarter of the course), we did many similar exercises. We got a pretty good understanding of how conditions specify scenarios. At the end of the unit, we took Module A Comprehensive Exercise home and filled in a self-evaluation form shown below. It was the first and the last course during my college years that I had an opportunity to self-evaluate my own performance. As Prof. Fovia handed out the exercise, he pointed out a known fact: strong students tend to under-estimate and weak students tend to over-estimate their performance. It was sort of understandable. I tried to as accurate as I could.

That Weird Math Course Take-Home Exercise Self-Evaluation Form

Module	A
Your name	
Names of your collaborators	
List of exercises submitted on-time	Circle: 00 A1 A2 A3
List of exercises completed by this time	Circle: 00 A1 A2 A3
Approximate number of hours spent	hours (for all these exercises)
Self-evaluation (between 0 and 10)	
Adjustment by the instructor	

Module A Performance Goals (expected outcomes and abilities to be observed as a result of successful learning)

Note that "mathematical structures" (scenarios) and "logical statements" (conditions) are discussed only *informally*.

1. Model a variety of real-world phenomena as mathematical structures.
2. Analyze whether a mathematical structure satisfies a collection of logical statements.
3. Specify mathematical structures using logical statements.
4. Analyze, distinguish, and relate mathematical structures with respect to their components ~~and the properties associated with the components~~.
5. Identify cases where (i) different set of logical statements satisfy the same mathematical structures, and (ii) a set of logical statements satisfies multiple mathematical structures including unintended ones.
6. ~~Convince others that the modeling process is logically sound, using proofs and other methods of justification.~~

Justification referring to the performance goals:

Note: Coming up with a way to convince others (including the instructor) about your achievements can be seen as an application of Discrete Math. Carefully think about how you can do this *sufficiently* and *concisely* (in this course, the length of your writing is not important). If this appears difficult at first, you are expected to develop such a skill through the exchanges between you and the instructor. If you have questions, contact the instructor.

ππππππππππππππππππππππππππππππππππππππ

In Module A, we discussed the condition-scenario connection informally. As Prof. Fovia pointed out, we occasionally suffered from the ambiguity of the English language. To avoid the problem, he introduced formal representations in Module B (second quarter). In reality, the discussion in Module A already touched upon various mathematical notions, including sets, relations, functions, (mathematical) structures, and logic. I thought that it was cool to learn formal representations of all these concepts. However, I will not make a big deal about it in this essay. Except for real mathematicians, we all forget most of math symbols any way. More important thing is that the idea of condition-scenario connection still sticks to me ... to this date. So, I will try to share what I learned without going into the details of formal representations. One of the key exercise is shown below. Note that certain parts of the exercise which are not relevant here have been "grayed" (throughout this essay) and will be skipped.

Exercise 4: Crime Scene

Part 1

Not every crime scene is gruesome. You, as a detective, will investigate a potentially only troublesome case described as follows:

Information

1. There is/are footprint(s) on the deck.

2. If no cats visit the deck, there is no footprints on the deck.
3. If at least one cat visits the deck, there are more than one cats.
4. No human have seen cat(s) on the deck.
5. If a human visits the deck, s/he must see herself/himself on the deck.
6. There are at least as many humans as cats.

A. Represent **Information** 1 through 5 as statements in First-Order Logic (FOL). In addition to the standard symbols in FOL, use only the following symbols specific to the crime scene:

- Unary predicate/relation symbols: *cat*, *human*, *footPrintOnDeck*, *visitDeck*
- Binary predicate/relation symbol: *seeOnDeck*

Note: Exclude **Information** 6 as it involves some advanced formulation.

B. What can you conclude about the number of cats that visited the deck? Explain by referring to some of the proof techniques discussed in connection to Propositional Logic (ref. Unit B5).

Note: There are multiple crucial steps in this process. Identify all of them.

Here is my response. **Information 1** requires that there were footprints. But the existence of footprints contradicts the "if" part of **Information 2**. So, one or more cats must have visited the deck. Then, **Information 3** requires that there were **at least two cats**. Let's consider the case of two cats. Now, I need to check other part of **Information** is also met. **Information 6** requires that there are at least two humans. Let's consider two humans. They saw neither of the cats (**Information 4**). The humans are not required to visit the deck. But if they do, each of them must see her/himself (**Information 5**).

To summarize, here is the list of objects, a property, and a relation:

- Cats (objects): *c1*, *c2*

- Humans (objects): *h1*, *h2*
- FootPrints (objects): *f*
- VisitDeck (property): *c1*
- SeeOnDeck (relation): no instance

C. Suppose that you need to check all the pairs of humans and cats in the crime scene (and you do not even know the entire sets). How would you conclude whether a human saw a cat? Identify the relevant statement(s) in **Information**, and analyze the corresponding FOL statement using some technique for dealing with quantifiers and negation.

Based on **Information 4**, no humans saw a cat on the deck. But they could have seen a cat elsewhere.

Next, consider a structure **CrimeScene** = (*Objects*, *Cats*, *People*, *FootPrintsOnDeck*, *VisitDeck*, *SeeOnDeck*) where

1. *Objects* contains all the objects involved in the crime scene.
2. *Cats* and *People* are subsets of *Objects*, and interpret the unary predicate symbols *cat* and *human*, respectively. For example, *human*(*a*) is true if and only if an object $a \in People$.
3. *FootPrintsOnDeck* defines the meaning of the predicate symbol *footPrintOnDeck*. For example, *footPrintOnDeck* (*a*) is true if and only if $a \in FootPrintsOnDeck$, i.e., *a* is a foot print on the deck. Note that $FootPrintsOnDeck \subseteq Objects$.
4. *VisitDeck* defines the meaning of the predicate symbol *visitDeck*, possibly applicable to both Santa Claus and reindeer. For example, *visitDeck*(*a*) is true if $a \in VisitDeck$, i.e., *a* visited the deck. Note that $VisitDeck \subseteq Objects$.
5. *SeeOnDeck* defines the meaning of the predicate symbol *seeOnDeck*. For example, *seeOnDeck*(*a*, *b*) is true if and only if $(a, b) \in SeeOnDeck$, i.e., *a* has seen *b* on the deck.

D. Do all the cats, if any, need to have visited the deck? Explain.

There are no statements that exclude the possibility of a cat not visiting the deck. So, other than the required one cat, which must have visited the deck, others are not so required.

E. Could any human have visited the deck? Explain.

If they do, they must see themselves (**Information 5**). But they do not necessarily see a cat. So, they could have been there.

F. Formally define all the structure components of the *smallest* instance of **CrimeScene** that would satisfy all the statements in **Information** shown above. For relations/functions, **give their types** as well. Explain how you came to that conclusion.

Note: The smallest instance would include the minimal number of objects.

Part 2

As a detective, you know that many crime scenes follow similar patterns. So, you conduct a search and found a file on another case with the following information:

Alt-Information
1. There is/are footprint(s) on the deck.
2. If no youngsters visit the deck, there is no footprints on the deck.
3. If at least one youngster visits the deck, there are more than one youngsters.
4. No human have seen youngster(s) on the deck.
5. If a human visits the deck, s/he must see herself/himself on the deck.
6. There are at least as many humans as youngsters.

For this case, consider a structure **CrimeScene**$_2$ = (*Objects*, *Youngsters*, *People*, *FootPrintsOnDeck*, *VisitDeck*, *SeeOnDeck*) where the structure components are defined in a way similar to **Part 1**, except the following points:
- Instead of *cat* in **Part 1**, another unary predicate/relation symbol *youngster* is used.
- Instead of *Cats*, *Youngsters* is used as a subset of *Objects*, and interprets the unary predicate symbols *youngster*.

G. Compare **Part 1** and **Part 2** very carefully. What can you conclude?

Basically, the only difference is that *cat* is replaced with *youngster*. Although it didn't strike me immediately, I noticed that "youngster" can mean a youngster of *any* animal, including a human and a cat. So, this seems to introduce a possibility that youngsters are also humans but not vice versa.

Now, we know that at least one youngster visited the deck (**Alt-Information 1 & 2**). But if so, this youngster, being also a human, must have seen herself (**Alt-Information 5**). This contradicts **Alt-Information 4**. Then, these conditions don't seem to be satisfiable. There seem to be no scenario that would satisfy them.

Part 2 seemed odd to me. I didn't really understand the point of it.

ππππππππππππππππππππππππππππππππππππππ

The course went on. We learned various things, including formal representations, various types of "mathematical" scenarios (structures), and proof techniques. In the end, I couldn't believe how well I was able to read and write formal representations. I also became a little more aware of various situations where condition-scenario connections can be observed. For example, my daily activities are fully constrained by a set of conditions. Only satisfiable scenarios can take place. Nevertheless, there still are many possible scenarios and there still are many different ways of setting up conditions.

At the end of the course, there was a take-home Comprehensive Exercise. One of the problems was as follows:

Exercise 5: Slime Scene

Some scientists observed that the following obscure scientific law applies to the small amount of slime (certain sticky matter, which a lay person would not bother to analyze) found in their lab.

Law

1. $\exists x\, f(x)$

OK. This is technical. So, I will add my explanation. The statement means that there is at least one object *x* that satisfies the property *f*.

2. $\left(\neg\exists x\, (c(x) \wedge v(x))\right) \rightarrow \left(\neg\exists x\, f(x)\right)$

If there is no object *x* that satisfies both the properties *c* and *v*, then there is no object that satisfies the property *f*.

3. $\left(\exists x\, (c(x) \wedge v(x))\right) \rightarrow \left(\exists x \exists y\, (c(x) \wedge c(y) \wedge x \neq y)\right)$

If there is at least one object x that satisfies both the properties c and v, then there are at least one x and at least one y such that both x and y satisfy the property c and x and y are not the same.

4. $\neg\exists x\exists y\ (h(x) \wedge c(y) \wedge s(x, y))$

There are neither object x nor object y such that x satisfies the property h, y satisfies the property c, and x and y, in that order, satisfy the relation s.

5. $\forall x\ ((h(x) \wedge v(x)) \rightarrow s(x, x))$

For any object x, if x satisfies both the properties h and v, x satisfies the relation s on itself.

This **Law** is expected to explain the composition of the slime, which can be represented as a structure **Slime** = (O, C, H, F, V, S) where

- O contains all the involved objects.
- C and H are subsets of O, and interpret the unary predicate symbols c and h, respectively. For example, $h(a)$ is true if and only if $a \in H$.
- F defines the meaning of the unary predicate symbol f. For example, $f(a)$ is true if and only if $a \in F$.
- V defines the meaning of the unary predicate symbol v. For example, $v(a)$ is true if $a \in V$.
- S defines the meaning of the binary predicate symbol s. For example, $s(a, b)$ is true if and only if $(a, b) \in S$.

Since the actual exercise was a little technical, again, I add my explanation.

- First, the scenario in discussion involves some objects. Let's call the collection of the objects O.
- Within this collection O, there are sub-collections called C and P. For example, the sub-collection C consists of all and only objects that satisfy the property c. Analogously, the sub-collection H consists of all and only objects that satisfy the property h.
- There is another collection called F. F consists of all and only objects that satisfy the property f.
- There is another collection called V. V consists of all and only objects that satisfy the property v.

- There is another collection called *S*. *F* consists of all and only pairs of objects that satisfy the relation *s*. That is, the pair (*a*, *b*) is in *S* exactly when *a* and *b* satisfies the relation s in that order.

In addition, the scientists also noted that the structure satisfies the following fact as well.
Fact: $|C| \leq |H|$

This means that the size of the collection *C* is smaller than or equal to the size of *H*.

Now, we see the logic-structure [condition-scenario] connection. However, we do not know what exactly the predicate symbols (in the logic) and the corresponding relation symbols (in the structure) [all the symbols] mean. Thus, we should not introduce additional assumptions beyond what the **Law** and **Fact** specify. For example, you do not know whether the sets *C* and *P* intersect or are disjoint.

A. Formally define all the structure components of the *smallest* instance of **Slime** (call it $\textbf{Slime}_0$) that would satisfy all the statements in **Law** and the **Fact** shown above. For relations/functions, give their types as well. Explain how you came to that conclusion.

   Note: The smallest instance would include the minimal number of objects.

At this point, I want to come up with a scenario that would satisfy **Law** and **Fact**. I will also try to come up with the smallest such scenario. First, there must be one object, say *o1*, that satisfies the property *f* (**Law 1**). Since there is an object that satisfies *f*, there must be an object that satisfies both the property *c* and the property *v* (**Law 2**). I label this object *o2*. Since *o2* satisfies both the property *c* and the property *v*, there must be another object, say *o3*, that also satisfies both the property *c* and the property *v* (**Law 3**). Since there are two objects, *o2* and *o3*, that satisfy c, there must be at least two objects that satisfy the property *h* (**Fact**). Let's call these *o4* and *o5*. Neither of objects *o4* or *o5* can satisfy the relation *s* with neither of objects *o2* or *o3* (**Law 4**). Finally, there is no need for any object to satisfy both the property *h* and the property *v* (**Law 5**).

In summary, there are five objects that satisfy the properties and relation as follows:
- *f*: *o1*
- *c*: *o2*, *o3*
- *v*: *o2*, *o3*
- *h*: *o4*, *o5*
- *s*: none

If you developed detective skills in "Crime Scene" (Module B Comprehensive Exercise 2), you must have noticed some connection between Crime Scene and Slime Scene. In fact, although not many people would notice, the ability to analyze the logic-structure [condition-scenario] connection is an essential skill for detectives and scientists alike. For example, you may have realized that the statements in **Law** (formal) correspond to **Information** 1 through 5 (informal) in Crime Scene (in the given order), and the **Fact** correspond to **Information** 6.

Well, yes, I realized the correspondence. **Crime Scene** and **Slime Scene** have basically identical conditions and thus, identical scenarios must satisfy those conditions. I wasn't sure what was the big deal of introducing **Slime Scene** in this end-of-the-semester exercise.

Eventually, the course ended. As promised, Prof. Fovia gave us no tests. So, I was pretty happy about it. Despite (or thanks to) that, I felt that I learned quite a bit. I got used to thinking about condition-scenario connections regularly.

After the semester, I worked for the college's IT department as a help desk staff. One day, I noticed that Tara was also staffing the desk. I remember her from the Disc Math class. We formed a pair a few times during the semester. She was actually an Engineering student but minored in Computer Science. Since there were nobody asking for help, we started to talk about the Disc Math course. When I mentioned that the **Slime Scene** exercise was pointless, Tara objected. This is what I discovered from Tara.

In the **Slime Scene** exercise, **Law** was represented formally. Thus, there was no "additional" meaning associated with each symbol. Although I noticed the correspondence, e.g., the property *c* for being "cat," there was nothing to introduce additional, "extraneous" meaning to those symbols. So, for example, when we introduced the objects *o4* and *o5* that satisfy the property *h*, there was no reason to assume that they are distinct from the objects *o2* and *o3*. Suppose that the objects *o4* and *o5* are identical to the objects *o2* and *o3*, respectively. Then, there are only three objects: *o1*, *o2*, and *o3*. Now, one problem is that **Law 5** requires that *o2* and *o3* must satisfy the relation *s* on themselves, i.e., (*o2*, *o2*) and (*o3*, *o3*). But **Law 4** requires that if *o2* satisfies the properties *h* and *v*, (*o2*, *o2*) cannot hold. One way out of this would be to let only *o3* satisfy the property *h* as well. This way, *o2* can satisfy the property *v* but doesn't need to satisfy (*o2*, *o2*). In the mean time, *o3* can satisfy the property *h* as well but does not need to satisfy the property *v*. This scenario requires only four objects. In the **Crime Scene** context, this translates to introducing one "cat person." Of course, this was not a *realistic* idea in that context. But actually, this is only because of our "assumption" that "cats" (in the Exercise) are not "humans" (in the Exercise). There was nothing in the formal representation to exclude that possibility.

Furthermore, Tara pointed out that either *o2* or *o3* can also satisfy the property *f*. In the **Crime Scene** context, this translates to equate one of the cats to their footprints. Again, this is certainly a nonsensical scenario. But again, this is nonsense *only if* we introduce an additional "assumption" associated with the English language.

The whole point of all these exercises must have been to clarify the potential issues involved in condition-scenario connections. In this respect, I didn't really get what Prof. Fovia intended. At the beginning of the following semester, I had a chance to talk to Prof. Fovia. When I mentioned that Tara told me what she found out, Prof. Fovia said, "Sure, Tara got it perfectly. She was the only one. She even figured out how my jewelry-making experience made me a mathematician. But don't be discouraged. Condition-scenario connections are everywhere, all the time. You will see them when you need them."

ππππππππππππππππππππππππππππππππππππππ

That was more than ten years ago. Prof. Fovia is no longer teaching at the college or at any other institution either. I heard that he quit all of sudden ... in the middle of an interview by a radio station host. But I am still in touch with Prof. Fovia. That's nothing to do with his Disc Math, though. Whenever he goes on a trip, I am the one who takes care of his cat. I know why Prof. Fovia created that **Crime Scene** exercise for the first place.

By the way, I recently found Prof. Fovia's old essay about Discrete Math still floating on the Internet. Here it is.

Some Thoughts on Teaching Discrete Mathematics

Note: This essay was written on behalf of a Computer Science program.

1. Background and Motivations

It has been pointed out to me that Discrete Mathematics is a difficult course to teach. I was told that the materials are too abstract and dry for most students. In addition, many students seem to perceive the list of topics very disconnected. As a person who enjoys Discrete Math, this is a rather sad situation. We can say, "Discrete Math will be extremely useful when you learn more advanced subjects in Computer Science." This does not seem to convince and motivate most students. The discussion on what to cover in a Discrete Math course and what kind of examples are more effective is an important one. However, the focus of this essay will be on how to teach Discrete Math, esp., how to connect the topics, hoping that more students are convinced of the usefulness of Discrete Math *very early in the course*.

At my institution, there is an on-going college-wide transformation of all the programs and curricula into more learning-centered ones (cf. teaching-centered ones). The main action I tried in the past few years is to re-examine and align the learning goals, student assessment, and learning activities of my courses (regrettably, my older syllabi failed to integrate assessment tools that would directly check learning goals). So, I am inclined to discuss teaching Discrete Math also in the context of the transformative change.

2. My Approach: Discrete Math for Formal Modeling of the Real World

It is not my intention to insist or defend the following aspect as the core of Computer Science: "to transform real-world problems into computational ones and solve the computational problems" (such a discussion would belong elsewhere). However, I wanted to start from this general property of Computer Science which can be used in designing a Discrete Math course (in the spirit of learning-centered approach) and also wanted to emphasize the connection between computation and the real world.

The way I perceive Discrete Math is a means to model real world, formally. That is, I view Discrete Math as the first step of transforming real-world problems. This places Discrete Math at a unique position in a Computer Science curriculum, in contrast to many other courses which focus on solving computational problems. When students want to write a program to solve some real-world problem, they will need to apply Discrete Math at some point. For example, I asked my students, "in order to write a program to give a driving directions (like the one on a GPS), how would you represent the necessary information? What about the game of musical chairs, social dance steps, or foreign policies?" At the heart of such a modeling process, I see the logic-structure [condition-scenario] connection (at a more abstract level than data structures). That is, the essence of mathematical modeling can be seen as specification of a (mathematical) structure involving sets, relations, and/or functions, through logical statements.

The idea of logic-structure connection is implicit in many areas; but it is rarely explicitly discussed (except mainly in mathematical logic). But there seem to be many advantages in placing the logic-structure connection at the heart of representing real-world objects/phenomena, as described below.

- The process can be applied to virtually any objects/phenomena. Thus, it is possible to create examples, exercises, and mini projects that are relevant to students' life.
- Since sets, relations, and functions are components of structures and they can be

specified by logical statements, all these basic components of Discrete Math will be in need every time students model something.

- The topic of algebraic structures fits naturally as a special case of structure, i.e., an operational structure with a single carrier/set with some logical conditions on the operations, e.g., $x + (y + z) = (x + y) + z$ for all x, y, z (in a certain set). More generally, any logical statements would specify a collection of structures; conversely, a structure satisfy (typically multiple) collections of logical statements.
- Pretty much all of other Discrete Math topics can be discussed in connection to this general scheme. For example, groups as an operational structure, orderings as relational structure, Boolean algebra as both operational and relation structures, graphs and trees with slightly different logical conditions, languages/automata as tuples, discrete probability involving sample space, etc.
- Even proofs can be viewed as a sequence of logical statements that is consistent with the intended collection of structures.
- Further more, logical specification is an important idea in software specification.

One aspect I tend to emphasize is that (i) there are in general multiple ways to specify a structure and (ii) a collection of logical statements generally have multiple satisfying structures. Although this could complicate the use of logic-structure connection, I believe it is important to emphasize that precise and correct representation of objects/phenomena is generally difficult (and in many cases impossible). This point seems to be useful for students to develop a critical attitude and appreciate good communication as well as insight into program specification. If the idea of logic-structure connection is taken seriously, then it can be considered as a bonding principle behind all the topics and may justify a Discrete Math course on its own (although I am not against the "just-in-time" approach to Discrete Math).

As for the organization of the materials, I adopt a "spiral" approach, starting from informal discussion of logic-structure connection (with no math symbols) and gradually introducing formal notations. Such an approach may not work well with the traditional course organization of materials divided into topics. However, the spiral approach seems to work well with the idea of logic-structure connection. In my course, the notion of "set" is visited multiple times at various points in the semester. Such a non-traditional way of introducing materials has some potential drawbacks. For instance, it is not very straightforward to use currently available textbooks. However, there is no textbook focusing on the logic-structure connection anyway. In addition, at the end of the semester, I received a few comments that the informal part of the course was vague. But I do not necessarily take those comments negatively. Those students must have appreciated the formal representation.

I think that students in general can understand and appreciate what we can do with the logic-structure connection. The overall response of the students has been positive. Naturally, there still are many areas I want to improve on. But I feel that this is probably a good start and an interesting approach for discussion.

3. Non-Exam-Type Assessment

I feel that exams are not the way I want to assess students. I tried to integrate students' self-evaluation as part of student assessment (with calibration by the instructor). It was difficult for multiple reasons. Overall, though, the experience was very positive. Not only I was able to confirm that this type of assessment is possible within the standard course organization, but also that the approach can encourage many students to realize where they are and try to achieve more. I will be pursuing to improve this aspect in my other courses.